

Convex bodies with many elliptic sections

Isaac Arelio

Luis Montejano

Abstract

We show in this paper that two normal elliptic sections through every point of the boundary of a smooth convex body essentially characterize an ellipsoid and furthermore, that four different pairwise non-tangent elliptic sections through every point of the C^2 -differentiable boundary of a convex body also essentially characterize an ellipsoid.

1 Introduction

Bianchi and Gruber [2] proved that the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if for every direction, continuously, we can choose a hyperplane which intersects $\text{bd}K$ in an ellipsoid. The proof of this result leads to the following characterization of the ellipsoid: Let $K \subset \mathbb{R}^3$ be a convex body and let $\alpha > 0$. If for every support line of K there is a plane H containing it whose intersection with $\text{bd}K$ is an ellipse of area at least α , then $\text{bd}K$ is an ellipsoid.

This characterization requires that for every $p \in \text{bd}K$ and every direction in the support plane of K at p there is a section of $\text{bd}K$ in that direction, containing p , that is an ellipse. The aim of this work is to give a characterization

of the ellipsoid where for every boundary point only a finite number of ellipses containing that point are required.

The sphere was characterized in this manner by Miyaoka and Takeuchi [5, 9] as the unique compact, simply connected C^∞ surface that satisfies one of the following properties: i) contains three circles through each point; ii) contains two transversal circles through each point; or iii) contains one circle inside a normal plane. In our paper, we shall show that two normal elliptic sections through every point of the boundary of a smooth convex body essentially characterize an ellipsoid and furthermore, four different pairwise nontangent, elliptic sections through every point of the C^2 -differentiable boundary of a convex body also characterize an ellipsoid.

The set

$$C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 + \frac{5}{4}xyz \leq 1, \max\{|x|, |y|, |z|\} \leq 1\}$$

[1, Example 2] is a convex body whose boundary is not an ellipsoid. The planes parallel to $x = 0$, $y = 0$ and $z = 0$ intersect $\text{bd}C$ in ellipses. Thus there are two ellipses for every point p in the boundary of C and there are three ellipses for every point p in the boundary of C except at six points.

Before giving the precise statement of the main theorems, we introduce the concepts and results we will use.

2 Definitions and auxiliary results

Let us consider a continuous function $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$. For every $p \in \text{bd}K$, let L_p be the line through p in the direction $\Psi(p)$. We say that Ψ is an *outward* function if for every $p \in \text{bd}K$,

- the line L_p is not tangent to $\text{bd}K$, and $\{p + t\Psi(p) | t > 0\}$ is not in K ; and
- there is a point $q \in \text{bd}K \setminus L_p$ such that the line L_q intersects the line L_p at a point of the interior of K .

For example, the function $\eta: \text{bd}K \rightarrow \mathbb{S}^2$ such that for every $p \in \text{bd}K$, $\eta(p)$ is the normal unit vector to $\text{bd}K$ at p is an outward function. To see this, let O be the midpoint of the interval $L_p \cap K$ and consider the farthest and the

nearest point of $\text{bd}K$ to the point $O \in \text{int}K$. One of them, call it q , is not in L_p , otherwise K would be a solid sphere, in which case the assertion is trivially true; hence clearly $O \in L_q$.

Let K be a convex body and let H_1 and H_2 be two planes. In the following it will be useful to have a criterion to know when two sections $H_1 \cap \text{bd}K$ and $H_2 \cap \text{bd}K$ intersect in exactly two points. This holds exactly when the line $L = H_1 \cap H_2$ intersects the interior of K .

We say that a collection of lines \mathcal{L} in \mathbb{R}^{n+1} is a *system of lines* if for every direction $u \in \mathbb{S}^n$ there is a unique line L_u in \mathcal{L} parallel to u .

Given a system of lines \mathcal{L} we define the function $\delta: \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ which assigns to every direction $u \in \mathbb{S}^n$ the point $\delta(u) \in L_u$ which traverses the distance from the origin to the line L_u . When δ is continuous we say that \mathcal{L} is a *continuous system of lines*. Finally, the set of intersections between any two different lines of the system will be the *center* of \mathcal{L} .

A continuous system of lines has a certain property that will be useful for our purpose which is stated in the following lemma.

Lemma 1. *Let \mathcal{L} be a continuous system of lines in \mathbb{R}^{n+1} . For every direction $u \in \mathbb{S}^n$, there exists $v \in \mathbb{S}^n$, $v \neq u$, such that the lines L_u and L_v have a common point.*

Proof. Let H be the plane through the origin that is orthogonal to L_u . Without loss of generality assume that H is the plane $z = 0$ and L_u contains the origin. Define \mathcal{L}_H as the set of orthogonal projections of all lines in \mathcal{L} parallel to H onto H ; \mathcal{L}_H is then a system of lines in H .

By [8, Proposition 3], there is a line in \mathcal{L}_H passing through the origin. This means that there is a direction v in H with the property that the line $L_v \in \mathcal{L}$ intersects L_u . \square

Suppose that there is a continuous system of lines \mathcal{L} such that the center of \mathcal{L} is contained in the interior of K . Note that every line of \mathcal{L} thus intersects the interior of K . For every $p \in \text{bd}K$, let L_p be the unique line of \mathcal{L} through p . Let us define the continuous function $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$ in such a way that $\Psi(p)$ is the unique unit vector parallel to L_p with the property that $\{p + t\Psi(p) | t > 0\}$

is not in K . By Lemma 1, $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$ is an outward function. If $K \subset \mathbb{R}^3$ is a strictly convex body, then the system of diametral lines of K is a continuous system of lines whose center is contained in the interior of K .

The following lemma will be of use to us in Section 4.

Lemma 2. *Let M_1 and M_2 be two surfaces tangent at $p \in M_1 \cap M_2$. If the normal sectional curvatures of M_1 and M_2 coincide in three different directions, then the normal sectional curvatures of M_1 and M_2 coincide in every direction.*

Proof. Suppose the Euler curvature formula for the first surface M_1 is given by

$$\kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta),$$

and for the second surface M_2 by

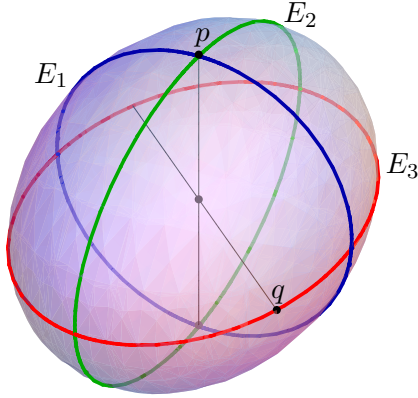
$$\kappa'_1 \cos^2(\theta - t_0) + \kappa'_2 \sin^2(\theta - t_0).$$

Suppose that the difference between these two expressions has three zeros. After simplifying the difference, replacing $\cos^2(x)$ and $\sin^2(x)$ by their corresponding expressions in $\cos(2x)$ and $\sin(2x)$, we obtain an expression of the form $A + B \cos(2\theta) + C \sin(2\theta)$, where A , B and C depend only on the principal curvatures and the angle t_0 . Since θ lies in the unit circle, where the number of zeros is bounded by the number of critical points, we may assume that the expression $C \cos(2\theta) - B \sin(2\theta)$ has at least three zeros. This implies that $A + B \cos(2\theta) + C \sin(2\theta) = 0$ and hence that the normal sectional curvatures of M_1 and M_2 coincide in every direction. \square

3 Two elliptic sections through a point

Theorem 3.1. *Let $K \subset \mathbb{R}^3$ be a convex body and α a given positive number. Suppose that there is an outward function $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$ such that for every $p \in \text{bd}K$ there are planes H_1, H_2 determining an angle at least α , where $H_1 \cap H_2$ is the line L_p through p in the direction $\Psi(p)$ and $K \cap H_i$ is an elliptic section, for $i = 1, 2$. Then $\text{bd}K$ is an ellipsoid.*

Proof. Let $p \in \text{bd}K$. By hypothesis there are two planes H_1 and H_2 such that $L_p = H_1 \cap H_2$ and $E_i = \text{bd}K \cap H_i$ is an ellipse, for $i = 1, 2$. Furthermore, there is a point $q \in \text{bd}K$ such that the line L_q intersects L_p at a point in the interior of K . Hence at least one of the elliptic sections through L_q is different from E_1 and E_2 ; call this section E_3 . We have that E_3 has two points in common with E_i , for $i = 1, 2$, because $L_p \cap L_q$ belongs to the interior of both sections.



We will now show that there is a quadric surface which contains E_i , for $i = 1, 2, 3$. Let

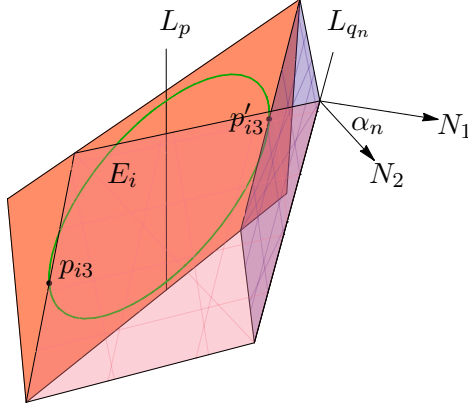
$$E_1 \cap E_2 = \{p, p'\}$$

and for $i = 1, 2$,

$$E_i \cap E_3 = \{p_{i3}, p'_{i3}\}.$$

We can choose $p_i \in \text{bd}E_i$ distinct from p, p', p_{i3} and p'_{i3} for $i = 1, 2$. Then the points $p, p', p_{13}, p'_{13}, p_{23}, p'_{23}, p_1, p_2$ and p_3 uniquely determine a quadric surface Q . Furthermore E_i has five points in common with Q . This implies that $E_i \subset Q$, for $i = 1, 2, 3$.

Now we will verify that there is an open neighborhood \mathcal{N} of p such that $\text{bd}K \cap \mathcal{N}$ is contained in Q . Suppose that there is no such neighborhood. Then there is a sequence $\{q_n\}_{n \in \mathbb{N}}$ in $\text{bd}K \setminus Q$ such that $\lim_{n \rightarrow \infty} q_n = p$, and moreover, the lines L_{q_n} converge to the line L_p . Our strategy is to prove that if n is sufficiently big, then one of the elliptic sections of K through L_{q_n} intersects each elliptic section E_i at two points. It is clear that if n is sufficiently big, both elliptic sections of K through L_{q_n} intersect E_3 at two points, because L_{q_n} intersects the interior of the ellipse E_3 . The same holds if L_{q_n} intersects the interior of the ellipse E_i . Suppose then that L_{q_n} does not intersect the interior of the ellipse E_i , and let N_1 be the unit vector normal to the plane determined by L_{q_n} and p_{i3} in the direction of the semiplane not containing p'_{i3} , and let N_2 be the unit vector normal to the plane determined by L_n and p_{i3} in the direction of the semiplane not containing p_{i3} . Let α_n be the angle determined by N_1 and N_2 .



The convergence of $\{L_{q_n}\}_{n \in \mathbb{N}}$ to L_p implies that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then there is $k_i \in \mathbb{N}$ such that for every $n \geq k_i$, we have $\alpha_n < \alpha$. This means that at least one of the normal elliptic sections through L_{q_n} intersects the relative interior of the segment between p_{i3} and p'_{i3} and therefore has two points in common with E_i .

Thus there is $k_0 \in \mathbb{N}$ such that for $n > k_0$ there is a plane H_n through L_{q_n} which determines an elliptic section $F_n = H_n \cap \text{bd}K$ and F_n has 6 points in common with Q . This shows that F_n coincides with the conic $H_n \cap Q$, and hence that $q_n \in Q$. This proves that there is an open neighborhood \mathcal{N} of p such that $\mathcal{N} \cap \text{bd}K \subset Q$.

We conclude that there is a finite open cover of $\text{bd}K$ in which every element coincides with a quadric; by the connectedness of $\text{bd}K$ and the fact that two quadrics that coincide in a relative open set of $\text{bd}K$ must be the same quadric, $\text{bd}K$ is thus contained in a quadric. Therefore $\text{bd}K$ is an ellipsoid. \square

Theorem 3.2. *Let $K \subset \mathbb{R}^3$ be a convex body and α a given positive number. Suppose that there is a continuous system of lines \mathcal{L} such that the center of \mathcal{L} is contained in the interior of K and through every L in \mathcal{L} there are planes H_1, H_2 determining an angle at least α such that $K \cap H_i$ is an elliptic section, for $i = 1, 2$. Then $\text{bd}K$ is an ellipsoid. Moreover, if for every L in \mathcal{L} one of the elliptic sections is a circle, then $\text{bd}K$ is a sphere.*

Proof. Note first that every line of \mathcal{L} intersects the interior of K . For every $p \in \text{bd}K$, let L_p be the unique line of \mathcal{L} through p and let us define the continuous function $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$ in such a way that $\Psi(p)$ is the unique unit vector parallel to L_p with the property that $\{p + t\Psi(p) | t > 0\}$ is not in K . By Lemma 1, $\Psi: \text{bd}K \rightarrow \mathbb{S}^2$ is an outward function and by Theorem 3.1, $\text{bd}K$ is an ellipsoid.

Let L_0 in \mathcal{L} be the line parallel to the diameter of K . Since there is a circular section of K through L_0 parallel to the diameter, and since every section of the ellipsoid $\text{bd}K$ parallel to this circular section is also circular, then there is a circular section of the ellipsoid $\text{bd}K$ through the diameter. This implies that the three axes of the ellipsoid $\text{bd}K$ have the same length and therefore that $\text{bd}K$ is a sphere. \square

Let $K \subset \mathbb{R}^3$ be a smooth, strictly convex body and let $p \in \text{bd}K$. Suppose that H is a plane through the unit normal vector of K at p . Then we say that the section $H \cap K$ is a *normal section* of K at p . If H is a plane containing

the diametral line of K through p , then we say that the section $H \cap K$ is a *diametral section* of K at p .

Theorem 3.3. *Let $K \subset \mathbb{R}^3$ be a smooth, strictly convex body and α a given positive number. Suppose that through every $p \in \text{bd}K$ there are two elliptic normal (respectively diametral) sections determining an angle at least α . Then $\text{bd}K$ is an ellipsoid.*

Motivated by the fact that for every diametral line of an ellipsoid there are two sections of the same area, we have the following result.

Corollary 3.4. *Let $K \subset \mathbb{R}^3$ be a smooth, strictly convex body and α a given positive number. Suppose that through every diametral line there are three elliptic sections of the same area determining an angle at least α . Then $\text{bd}K$ is a sphere.*

Proof. By Theorem 3.3, $\text{bd}K$ is an ellipsoid. Note now that the hypothesis implies that the section through the center of K orthogonal to one of the axes is a circle. This implies that the three axes of the ellipsoid $\text{bd}K$ have the same length. \square

4 Four elliptic sections through a point

Theorem 1. *Let $K \subset \mathbb{R}^3$ be a convex body with a C^2 -differentiable boundary and let $\alpha > 0$. Suppose that through every $p \in \text{bd}K$ there are four planes $H_{p_1}, H_{p_2}, H_{p_3}, H_{p_4}$ satisfying:*

- $H_{p_j} \cap \text{bd}K$ is an ellipse E_{p_j} of area greater than $\alpha > 0$, $j = 1, 2, 3, 4$,
- for $1 \leq i < j \leq 4$, the ellipses E_{p_i} and E_{p_j} are not tangent.

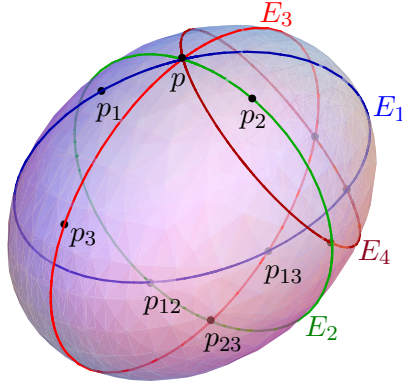
Then $\text{bd}K$ is an ellipsoid.

Proof. We shall prove that locally, $\text{bd}K$ is a quadric. Let $p \in \text{bd}K$. Then the line $L_{ij} = H_{p_i} \cap H_{p_j}$ intersects the interior of K , because the ellipses E_{p_i} and E_{p_j} are not tangent. For $1 \leq i < j \leq 4$, let $\{p, p_{ij}\} = E_{p_i} \cap E_{p_j} = L_{ij} \cap \text{bd}K$. We shall show that there is a quadric Q that contains the four ellipses $E_{p_1},$

$E_{p_2}, E_{p_3}, E_{p_4}$ and from that, we shall prove that Q coincides with $\text{bd}K$ in a neighborhood of p .

In order to prove that $E_{p_1}, E_{p_2}, E_{p_3}, E_{p_4}$ are contained in a quadric Q , we follow the spirit of Gruber and Bianchi in [2]. If $p \in \text{bd}K$, let H_p be the support plane of K at p and let $L_{p_i} = H_{p_i} \cap H_p, i = 1, \dots, 4$.

Case a). No three of the planes $E_{p_1}, E_{p_2}, E_{p_3}, E_{p_4}$ share a line. Then we have six distinct points $p_{ij} \in \text{bd}K$. By elementary projective geometry, let Q be the unique quadric which is tangent to the plane H_p at p and contains the six points $\{p_{ij} \mid 1 \leq i < j \leq 4\}$. Furthermore, let $F_1 \subset H_1$ be the unique quadric which is tangent to the line L_{p_1} at p and contains the three points $p_{12}, p_{13}, p_{1,4}$. Then $Q \cap H_{p_1} = F_1 = E_{p_1}$. Similarly, the ellipses E_{p_2}, E_{p_3} , and E_{p_4} are contained in the quadric Q .



Case b). Three of planes $H_{p_1}, H_{p_2}, H_{p_3}, H_{p_4}$ share a line. So without loss of generality, suppose $H_{p_1} \cap H_{p_2} \cap H_{p_3} = L_{12} = L_{13} = L_{23}$ and $p_{12} = p_{13} = p_{23}$. Let Γ be the supporting plane of K at $p_{12} = p_{13} = p_{23}$ and let Q be the unique quadric tangent to H_p at p tangent to Γ at $p_{12} = p_{13} = p_{23}$ that contains arbitrarily chosen points $p_i \in E_i \setminus \{p, p_{12} = p_{13} = p_{23}\}$. Then $Q \cap H_1$ is the unique quadric contained in H_1 , tangent to L_{p_1} at p , tangent to $\Gamma \cap H_1$

at $p_{12} = p_{13} = p_{23}$ and that contains p_1 . This implies that the ellipse E_1 is contained in Q . Similarly, $E_2 \cup E_3 \subset Q$. Let F_4 be the unique quadric contained in E_4 , tangent to L_{p_4} at p , and that contains the three distinct points p_{14} , p_{24} , and p_{34} . Clearly $F_4 = Q \cap H_4$ but also $F_4 = E_4$, which implies that $E_4 \subset Q$.

Case c). The four planes $H_{p_1}, H_{p_2}, H_{p_3}, H_{p_4}$ share a line. Let $H_{p_1} \cap H_{p_2} \cap H_{p_3} \cap H_{p_4} = L_0$ and $L_0 \cap \text{bd}K = \{p, R\}$. At $R \in \text{bd}K$ there is only one support plane H_R of K . By using a projective homeomorphism if necessary, we may assume without loss of generality that H_p is parallel to H_R and also that L_0 is orthogonal to both H_p and H_R . As in Case b), there is a quadric Q containing $E_{p_1}, E_{p_2}, E_{p_3}$. By Lemma 2, we know that the sectional curvature at three different directions determines the sectional curvature at all other directions. In our situation, this implies that the curvature of the two ellipses $Q \cap H_{p_4}$ and E_4 are the same. This, together with the fact that both ellipses $Q \cap H_{p_4}$ and E_4 contained in H_4 are tangent at p to $H_4 \cap H_p$ and tangent at R to $H_4 \cap H_R$, implies that $Q \cap H_{p_4} = E_4$ and hence that $E_4 \subset Q$.

We are ready to prove that there is a neighborhood U of $\text{bd}K$ at p such that $U \subset Q$. Suppose this is not so; then there is a sequence $q_1, q_2, \dots, \in \text{bd}K \setminus Q$ converging to p . For every $q_i \in \text{bd}K$, let H_{q_i} be a plane through q_i such that $H_{q_i} \cap \text{bd}K$ is an ellipse E_{q_i} of area greater than α . By considering a subsequence and renumbering if necessary, we may assume that the H_{q_i} converge to a plane H through p . The fact that $H_{q_i} \cap \text{bd}K$ is an ellipse of area greater than $\alpha > 0$ implies that H is not a tangent plane of K . So let $L = H \cap H_p$. We may assume without loss of generality that $L \neq L_{p_1}, L_{p_2}, L_{p_3}$, so the plane H intersects each of the ellipses E_1, E_2, E_3 at two points, one of them being p . Since H_{q_i} converges to H , we may choose n_0 such that if $i > n_0$, then H_{q_i} intersects the ellipse E_j at two distinct points, $j = 1, 2, 3$. Therefore, since the quadrics E_{q_i} and $Q \cap H_{q_i}$ share at least six points, they should be the same. This implies that $q_i \in Q$ for $i > n_0$, contradicting the fact that $q_1, q_2, \dots, \in \text{bd}K \setminus Q$.

The fact that $\text{bd}K$ is compact, connected and locally a quadric implies that $\text{bd}K$ is an ellipsoid. \square

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